Summary (Part 1 Pure)

Arithmetic

Definitions;
Sum = One number plus another
Difference = One number minus another
Product = One number times another
Quotient = One number divided by another
A number is the Product of its Factors
Primes are Numbers with no Factors except 1 and itself
HCF (Highest Common Factor) = Highest Factor that is Common to all numbers of a group
LCM (Lowest Common Multiplier) = Lowest number that has all numbers of a group as Factors
Numerator is the Number at top of Fraction
Denominator is the Number at bottom of Fraction
Reciprocal = 1 Divided by the Number
Factorial is the Product of all Numbers from 1 to the Number and is written with ! For example 4! = 1 x 2 x 3 x 4
Ratio is the Comparison of 2 or more Numbers. For example 15 : 5 has the same Ratio as 3 : 1
Square of a Number = Number times itself, written as N^2. For example $5^2 = 25$
Square Root of a Number times itself = The Number. Square Root is written as $\sqrt{N}$. For example $\sqrt{25} = \pm 5$
Index, or Power = Number of times a Number is multiplied by itself. For example $5^3 = 5 \times 5 \times 5$ has the index of 3
Scientific Notation = Number expressed as a number between 0 and 10 times powers of 10
Binary = Number expressed in 2 digits (0 & 1)
Octal = Number expressed in 8 digits (0 - 7)
Hexadecimal = Number expressed in 16 digits (0 - 9 and A - F)

Hex(abcd) = Decimal $(d + c \times 16 + b \times 16^2 + a \times 16^3)$
$2^{10} = 1024$ $\quad \sqrt{2} \approx \pm 1.414$ $\quad 1/\sqrt{2} \approx \pm 0.707$ $\quad \sqrt{3} \approx \pm 1.732$ $\quad \sqrt{10} \approx \pm 3.16$

Algebra

Definitions;
Irrational functions are Functions that contain a square root, or cube root etc. Rationalised Functions do not.
Equations are statements that two functions are equal
Simultaneous equations are a set of Equations connecting two or more unknowns
Greek letter sigma Σ means "Sum of Terms Like"
Coefficient. For example in the function $7x^2 – 5x + 3$, the coefficient of x^2 is 7and the coefficient of x is –5

$(– a)$ times $(– b) = + ab$
a^m times $a^n = a^{m+n}$
$(a^m)^n = a^{mn}$
$a^0 = 1$ and $a^1 = a$ and $a^{-n} = 1/a^n$ and $a^{(1/n)} = \sqrt[n]{(a)}$
$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$
To factorize $ax^2 + bx + c$ — If ac (ie a times c) is -ive, look for factors of ac whose sum = b
If ac is +ive, look for factors whose difference = $\pm b$
$x^2 – a^2 = (x + a)(x – a)$
$(x \pm a)$ is a factor of $x^3 \pm a^3$ — Put a = 1 to get $(x \pm 1)$ is a factor of $x^3 \pm 1$
$a_0 + a_1x + a_2x^2 + ... + a_nx^n$ can be divided by $b_0 + b_1x + b_2x^2 + ... + b_{n-r}x^{n-r}$ to get Quotient and Remainder
Method is similar to Arithmetical Long Division. Include missing terms using zero as the coefficient
For example $(ax^2 + bx + c)/(x + d) = [ax + (b – ad)] + [(c – bd + ad^2)/(x + d)]$
Divide F(x) by (x – a) and the Remainder is F(a)
$F(x) / [(x + \alpha_1)(x + \alpha_2)(x + \alpha_3)] = A_1/(x + \alpha_1) + A_2/(x + \alpha_2) + A_3/(x + \alpha_3)$ where $A_1 = F(–\alpha_1) / [(\alpha_2 – \alpha_1)(\alpha_3 – \alpha_1)]$ etc
If the Numerator is same or higher power than the Denominator, divide it by the Denominator first.
In each fraction, the Numerator contains x to one power less than the Denominator
Two equal factors $F(x) / [(x + \alpha_1)^2(x + \alpha_3)] = A_1/(x + \alpha_1) + A_2/(x + \alpha_1)^2 + A_3/(x + \alpha_3)$
$1/(a + \sqrt{b}) = (a – \sqrt{b})/(a^2 – b)$ and $1/(a – \sqrt{b}) = (a + \sqrt{b})/(a^2 – b)$ this puts the irrational term in the numerator
$i^2 = – 1$
$(a + ib)(a – ib) = a^2 + b^2$ and $1/(a + ib) = (a – ib)/(a^2 + b^2)$ and $1/(a – ib) = (a + ib)/(a^2 + b^2)$. Puts in form c + i d
Solution to the quadratic $Ax^2 + Bx + C = 0$ is $x = [– B \pm \sqrt{(B^2 – 4AC)}] / 2A$
$\alpha_1 + \alpha_2 = – B/A$ and $\alpha_1 \alpha_2 = C/A$ where α_1 are α_2 are the two solutions
If $4AC > B^2$ then the two solutions are a conjugate pair, $\alpha + i\beta$ and $\alpha – i\beta$

An equation has as many solutions as the highest power of x after rationalizing. A quadratic has 2, a cubic has 3.

Geometry

Definitions;
Angles, one revolution is 360 degrees = 2 π radians, a Right Angle is 90 degrees = π /2 radians
Equilateral Triangle has all sides equal
Isosceles Triangle has two angles equal
Similar Triangles are Two Triangles with same angles. The Ratio between the sides is the same in both.
Congruent Triangles are Two Triangles exactly the same
"Normal to", "Orthogonal to" and "Perpendicular to" mean "at right angles to"
Hypotenuse is the Side of a Triangle opposite a right angle
Tangent is a line that just touches a curve and is parallel to the curve at that point.

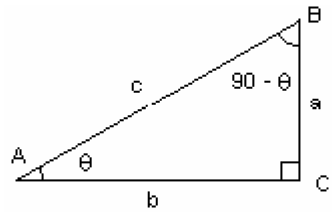
Definition of radian. Angle in radians = Length of arc of a circle divided by radius. For circle $s = r\,\theta$

Conditions for congruent triangles either (i) 3 sides same, or (ii) 2 sides and the angle between them the same, or (iii) 2 angles and a corresponding side the same or (iv) hypotenuse and one other side the same.

Triangle Sum of angles = 180^0
Area = (½) Base x Height
Pythagoras (for a Right Angled Triangle) $a^2 + b^2 = c^2$
Examples $3^2 + 4^2 = 5^2$, $6^2 + 8^2 = 10^2$, $12^2 + 5^2 = 13^2$
Medians meet at a point, so do Angle bisectors, so do lines from each apex perpendicular to opposite side, so do perpendiculars from mid points of sides

Circle Circumference = $\pi D = 2\pi R$ and Area = πR^2 where D is the Diameter and R the Radius
Rectangle or Parallelogram Area = Base x Height (Height is measured normal to base)

Trigonometry



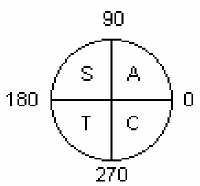
$\sin\theta = a/c$ and $\cos\theta = b/c$ and $\tan\theta = a/b$
$\text{Cosec}\,\theta = 1/\sin\theta$ and $\sec\theta = 1/\cos\theta$ and $\cot\theta = 1/\tan\theta$
$\sin\theta / \cos\theta = \tan\theta$

$\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$

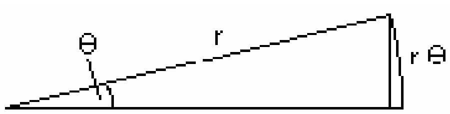
$\sin(90^0 - \theta) = \cos\theta$ and $\tan(90^0 - \theta) = \cot\theta$

$\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = +\cos\theta$ and $\tan(-\theta) = -\tan\theta$
$\sin(180^0 - \theta) = +\sin\theta$ and $\cos(180^0 - \theta) = -\cos\theta$ and $\tan(180^0 - \theta) = -\tan\theta$
$\sin(180^0 + \theta) = -\sin\theta$ and $\cos(180^0 + \theta) = -\cos\theta$ and $\tan(180^0 + \theta) = +\tan\theta$

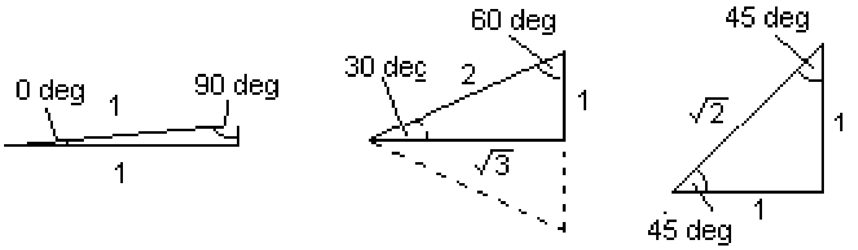
CAST Angles (-90^0 to 0^0) **C**os + ive, Sin and Tan – ive
Angles (0^0 to 90^0) **A**ll + ive
Angles (90^0 to 180^0) **S**in + ive, Cos and Tan – ive
Angles (180^0 to 270^0) **T**an + ive, Sin and Cos – ive



If θ is small and in radians then $\sin\theta = \theta$
and $\tan\theta = \theta$ and $\cos\theta = 1 - (½)\,\theta^2$



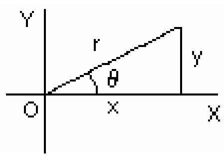
For other angles, see the diagrams



Sin (A + B) = Sin A Cos B + Cos A Sin B
Cos (A + B) = Cos A Cos B - Sin A Sin B
Tan (A + B) = (Tan A + Tan B) / (1 – Tan A Tan B)
Sin (2A) = 2 Sin A Cos A
$\text{Cos}(2A) = \text{Cos}^2A - \text{Sin}^2A$
$\text{Tan}(2A) = 2\text{Tan}A /(1 - \text{Tan}^2A)$
Sin A Cos B = (½) [Sin (A + B) + Sin (A – B)]
Cos A Cos B = (½) [Cos (A + B) + Cos (A – B)]
Sin A Sin B = (½) [Cos (A – B) – Cos (A + B)]
Sin A + Sin B = 2 Sin [(½) (A + B)] Cos [(½) (A – B)]
Sin A – Sin B = 2 Cos [(½) (A + B)] Sin [(½) (A – B)]
Cos A + Cos B = 2 Cos [(½) (A + B)] Cos [(½) (A – B)]
Cos A – Cos B = – 2 Sin [(½) (A + B)] Sin [(½) (A – B)]
$\text{Sin}^2A - \text{Sin}^2B = \text{Sin}(A + B)\ \text{Sin}(A - B)$
$\text{Cos}^2 A - \text{Cos}^2 B = -\text{Sin}(A + B)\ \text{Sin}(A - B)$
$\text{Cos}^2 A - \text{Sin}^2 B = \text{Cos}(A + B)\ \text{Cos}(A - B)$

Triangle
Area = (½) ab SinC and Area = √ [s(s – a)(s – b)(s – c)] where s = (½)(a + b + c)
a/SinA = b/SinB = c/SinC and $c^2 = a^2 + b^2 - 2ab\text{Cos}C$

Co-ordinate Geometry



Cartesian co-ordinates, points are shown by x and y
Polar co-ordinates, points are shown by r and θ
$r = \sqrt{(x^2 + y^2)}$ and Tan θ = y/x and x = r Cos θ and y = r Sin θ

Straight line, slope m — $y = mx + c$
Line through (x_1, y_1) and (x_2, y_2) — $y_1 = m x_1 + c$ and $y_2 = m x_2 + c$ Solve for m and c
Angle between two lines — $\text{Tan}\,\theta = (m_1 - m_2)/(1 + m_1m_2)$
2 lines cross orthogonally if — $m_1 m_2 = -1$
Circle, centre at origin — $x^2 + y^2 = a^2$
Circle, centre at (g,h), radius a — $(x - g)^2 + (y - h)^2 = a^2$
Ellipse, centre at origin — $x^2/a^2 + y^2/b^2 = 1$
Parabola — $y^2 = 4ax$
Hyperbola — $xy = c^2$ or $x^2/a^2 - y^2/b^2 = 1$

Logarithms

By definition; $\log_a m = x$ where $a^x = m$
Hence $\log_a m + \log_a n = \log_a(mn)$
$\log_a m - \log_a n = \log_a(m/n)$
$n \log_a m = \log_a m^n$
$\log_b m = \log_a m / \log_a b$

Binomial

$(x + a)^n = x^n + n a x^{n-1} + [n (n - 1) / 2!] a^2 x^{n-2} + [n (n - 1) (n - 2) / 3!] a^3 x^{n-3} ..... + n! / [(n-r)! r!] a^r x^{n-r} + ... + a^n$

Matrices

Data can be displayed and manipulated in short hand in the form of Matrices.

Eg $a_1 x + a_2 y + a_3 z + a_4 = 0$ can be written (and solved) in Matrix form

$b_1 x + b_2 y + b_3 z + b_4 = 0$

$c_1 x + c_2 y + c_3 z + c_4 = 0$

(add or subtract lines to get coefficients 2 and 3 = 0 to solve for x etc)

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix} = 0$$

Determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

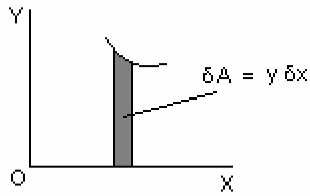
terms in sequence abc and numbers in sequence 123123 positive, others negative

Series

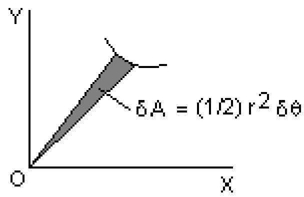
Definitions;
Arithmetical Progression AP is a Series with the same difference between all adjacent terms
Geometrical Progression GP is a Series with the same ratio between all adjacent terms

Sum S of AP, 1st term a, difference d, n terms. Add first term to last term, 2nd term to 2nd last etc
hence $2S = n[\{a\} + \{a + (n-1)d\}]$
Sum S of GP, 1st term a, ratio of terms p, n terms.
Then Series - p times Series = first term + last term hence $S = a(1 - p^n)/(1 - p)$
Sum of first n numbers is an AP $= n(n + 1)/2$
Sum of first n squares $= (1/6)n(n + 1)(2n + 1)$
Sum of first n cubes $= [(n + 1)n/2]^2$

Calculus



Definitions;
Differential of y, written dy/dx is Slope at x
Integral of y, written $\int y\,dx$ is the sum of areas of height y and width dx
$d/dx[ax^n] = a n x^{n-1}$ and $\int a x^n\,dx = a x^{n+1}/(n+1) + c$
Integration between limits. The value of the Integral with the final value of x minus its value with the initial x



In polar co-ordinates, $dy/dx = (\sin\theta\, dr/d\theta + r\cos\theta)/(\cos\theta\, dr/d\theta - r\sin\theta)$
In polar co-ordinates, Sum of areas $= \int (½)\, r^2\, d\theta$

Differential of a sum	$d/dx\,(u + v) = du/dx + dv/dx$
Differential of a product	$d/dx\,(u v) = v\, du/dx + u\, dv/dx$
Differential of a Fraction	$d/dx\,(u/v) = \{v\, du/dx - u\, dv/dx\}/v^2$
Differential with a change of variable	$dy/dx = dy/du \cdot du/dx$

ln is natural logarithm (ie Log to base e) where e = 1 + 1/1! + 1/2! + 1/3! + … to infinity
By definition, if $\ln(m) = x$ then $e^x = m$
$d/dx\,(e^x) = e^x$ and $\int (1/x)\,dx = \ln(x) + c$ and $d/dx\,[\ln(x)] = 1/x$
$a^x = e^{x \ln(a)}$

$d/dx\,(\sin x) = \cos x$ and $d/dx\,(\cos x) = -\sin x$ and $d/dx\,(\tan x) = \sec^2 x$

$d/dx\,\{\arcsin(x/a)\} = 1/\sqrt{(a^2 - x^2)}$ and $d/dx\,\{\arccos(x/a)\} = -1/\sqrt{(a^2 - x^2)}$
$d/dx\,\{\arctan(x/a)\} = a/(a^2 + x^2)$

MacLaurim's Theorem

Let $f(x) = a_0 + a_1x/1! + a_2x^2/2! + \ldots + a_rx^r/r! +$

Write $f_r(0)$ to mean r th differential of f(x) with x then made zero, hence $a_r = f_r(0)$

$f(x) = f(0) + f_1(0)\, x/1! + f_2(0)\, x^2/2! + f_3(0)\, x^3/3! + \ldots + f_r(0)\, x^r/r! +$

The series for many functions can be written down, eg

$\text{Sin } x = x - x^3/3! + x^5/5! - x^7/7! \ldots\ldots$

$\text{Cos } x = 1 - x^2/2! + x^4/4! - x^6/6! \ldots\ldots$

$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! \ldots\ldots$

$\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 \ldots\ldots$

Taylor's Theorem

$f(x) = f(a) + (x - a)f_1(a) + \ldots + [(x - a)^r / r!]f_r(a) + \ldots$

Hyperbolic Functions

Expand Cos nθ + i Sin nθ by MacLaurim's Theorem and the result is the expansion of $e^{in\theta}$

$\text{Cos } n\theta + i \text{ Sin } n\theta = e^{in\theta} = [\text{Cos } \theta + i \text{ Sin } \theta]^n$

$\text{Cos } \theta + i \text{ Sin } \theta = e^{i\theta}$ and $\text{Cos } \theta - i \text{ Sin } \theta = e^{-i\theta}$

$\text{Cos } \theta = \{e^{i\theta} + e^{-i\theta}\}/2$ and $\text{Sin } \theta = \{e^{i\theta} - e^{-i\theta}\}/2i$

By definition, Cosh and Sinh are these values of Cos and Sin without the i

$\text{Cosh } \theta = \{e^{\theta} + e^{-\theta}\}/2$ and $\text{Sinh } \theta = \{e^{\theta} - e^{-\theta}\}/2$

$\text{Tanh } \theta = (\text{Sinh } \theta)/(\text{Cosh } \theta)$ and $\text{Sech } \theta = 1/\text{Cosh } \theta$

$\text{Cosech } \theta = 1/\text{Sinh } \theta)$ and $\text{Coth } \theta = 1/\text{Tanh } \theta$

$\text{Cosh}^2 \theta - \text{Sinh}^2 \theta = 1$ and $1 - \text{Tanh}^2 \theta = \text{Sech}^2 \theta$

$\text{Sinh } 2x = 2 \text{ Sinh } x \text{ Cosh } x$ and $\text{Cosh } 2x = \text{Cosh}^2 x + \text{Sinh}^2 x$

$d/dx\, (\text{Sinh } x) = \text{Cosh } x$ and $d/dx\, (\text{Cosh } x) = \text{Sinh } x$ and $d/dx\, (\text{Tanh } x) = \text{Sech}^2 x$

Methods for Integration

In General Look for a substitution that will simplify the integral

$\int F(ax \pm b)dx$ indicates the substitution $u = (ax \pm b)$ thus du = a dx

eg $\int[\text{Sin}(x + a)]\, dx$ Put u = x + a thus Integral = $\int[\text{Sin}(u)]du = -\text{Cos}(u) + \text{constant}$

$\int[1/(x^2 + a^2)]dx$ indicates the substitution $x = a \text{ Tan}(u)$ and $dx = a \text{ Sec}^2(u)du$ or $x = a \text{ Sinh}(u)$ and $dx = a \text{ Cosh}(u)du$

eg $\int[1/(x^2 + a^2)]dx$ Put x = a Tan(u) this leads to $(1/a)\int du = u/a + \text{constant}$

Fractions If the denominator factorizes, Split into Partial Fractions;

$\int[1/\{(x \pm a)(x \pm b)\}]dx = \int[A/(x \pm a)]dx + \int[B/(x \pm b)]dx$

eg $\int[1/(x^2 - a^2)]dx = \int[(1/2a)/(x - a)]dx - \int[(1/2a)/(x + a)]dx = (1/2a)[\ln(x - a) - \ln(x + a)] + \text{constant}$

Integrals of Square Roots (Remember $1 - \text{Sin}^2 u = \text{Cos}^2 u$, $1 + \text{Sinh}^2 u = \text{Cosh}^2 u$, and $\text{Cosh}^2 u - 1 = \text{Sinh}^2 u$)

$\int[1/\sqrt{(a^2 - x^2)}]dx$ indicates the substitution $x = a \text{ Sin } (u)$ thus dx = a Cos (u) du

$\int[1/\sqrt{(a^2 - x^2)}]dx = \int[1/a\text{Cos}(u)]a\text{Cos}(u)du = \int du = u + \text{constant}$

$\int[1/\sqrt{(a^2 + x^2)}]dx$ indicates the substitution $x = a \text{ Sinh}(u)$ or $x = a \text{ Tan}(u)$

$\int[1/\sqrt{(x^2 + a^2)}]dx$ Put x = a Sinh(u) this leads to $\int du = u + \text{constant}$

$\int[1/\{\sqrt{(x^2 - a^2)}\}]dx$ indicates the substitution $x = a \text{ Cosh } (u)$

$\int[1/\{\sqrt{(x^2 - a^2)}\}]dx$ Put x = a Cosh u leads to $\int du = u + \text{constant}$

$\int[1/\{\sqrt{(ax^2 + bx + c)}\}]dx$ Remove the x term,

Put $a[(x + p)^2 + q] = ax^2 + bx + c$ Equate coefficients to solve for p and q, Put u = x + p and $r^2 = q$.

This leads to $(1/\sqrt{a})\int[1/\sqrt{(u^2 \pm r^2)}]du$ As above put u = r Sinh v or u = r Cosh v

$\int\sqrt{[(a + b\,x)/(c - d\,x)]}dx$ where a, b, c, and d are all +ive. Make coefficients of x = 1

$I = \sqrt{(b/d)}\int\sqrt{[(a/b + x)/(c/d - x)]}dx$. Put $x = p + q\text{Sin } \theta$ hence $dx = q\text{Cos } \theta\, d\theta$. Find values for p and q to get

$k\int\sqrt{[(1 + \text{Sin } \theta)/(1 - \text{Sin } \theta)]}\,\text{Cos } \theta d\theta$ Evaluate k. Multiply top and bottom by $(1 + \text{Sin } \theta)$ to get $k\int[1 + \text{Sin } \theta]d\theta$

Trigonometrical integrals

(i)Put in form $\int F(\text{Cos } x) \text{ Sin } x \, dx$, or $\int F(\text{Sin } x) \text{ Cos } x \, dx$ or $\int F(\text{Tan } x) \text{ Sec}^2 x \, dx$ and Substitute to get $\int F(u) \, du$

Similarly for hyperbolics Eg $\int \text{Sinh}^3 x \, dx = \int (\text{Cosh}^2 x - 1) \text{ Sinh } x \, dx = 1/3 \text{ Cosh}^3 x - \text{Cosh } x$ + constant

or (ii) Try the substitution $u = \text{Tan}(x)$ since $dx = du/(1 + u^2)$

or (iii) Try substituting $t = \text{Tan}(x/2)$ since $dx = 2 \, dt/(1 + t^2)$, $\sin(x) = 2t/(1 + t^2)$ and $\text{Cos}(x) = (1 - t^2)/(1 + t^2)$

$\int [1/(a \text{ Sin } x + b \text{ Cos } x + c)] dx$ indicates the substitution $t = \text{Tan}(x/2)$

$\int \text{Cos}^2(x) \, dx$ and $\int \text{Sin}^2(x) \, dx$ indicate the substitution $u = 2x$ since $\text{Cos}^2(x) = ½ [\text{Cos}(u) + 1]$ and $dx = (½) \, du$

1/D Method

The operator D is defined as d/dx. Thus D(y) = dy/dx hence $(D + a)(D + b)(y) = D^2(y) + (a + b)D(y) + ab \, y$

$D^{-1}(y) = \int y \, dx$

$D^n (e^{a x} V) = e^{a x} (D + a)^n V$

$[1/F(D)] e^{a x} = [1/F(a)] e^{a x}$

$F(D^2) \; (a \text{ Sin } mx + b \text{ Cos } mx) = F(-m^2) \; (a \text{ Sin } mx + b \text{ Cos } mx)$

$\int e^{ax} \text{Cos}(bx) \, dx$ and $\int e^{ax} \text{Sin}(bx)$ can be integrated by the 1/D method but it is simpler to consider the Real (or Complex) part of $\int e^{ax}[\text{Cos}(bx) + i \text{ Sin}(bx)] \, dx = \int e^{(a+ib)x} dx = [1/(a + ib)] e^{(a+ib)x}$ + constant

Integration by Parts

d/dx (u v) = v du/dx + u dv/dx, thus $\int u \, dv = uv - \int v \, du$ Use this formula to transform an Integral of a product

example (i) $\int x \text{ Sin } (x) \, dx$ Put x = u and Sin (x) dx = dv hence v = – Cos (x) and du = dx

example(ii) $\int x \ln (x) \, dx$ Put ln (x) = u and x dx = dv hence $v = (½) \, x^2$ and du = 1/x dx

Standard forms

y	dy / dx	$\int$ y dx
$a x^n$	$n a x^{n-1}$	$a x^{n+1} /(n + 1)$
a / x	$-a/x^2$	a ln(x)
Sin ω x	ω Cos ω x	(–1/ ω) Cos ω x
Cos ω x	– ω Sin ω x	(1/ ω) Sin ω x
Tan ω x	$\omega \text{ Sec}^2 \omega x$	– (1/ ω) ln (Cos ω x)
Sec x	tan x Sec x	ln (Sec x + Tan x)
Cosec x	– Cot x Cosec x	ln (Cosec x – Cot x)
Cot x	$-\text{Cosec}^2 x$	ln (Sin x)
Arc Sin (x/a)	$1 / \sqrt{(a^2 - x^2)}$	$X \text{ Arc Sin } (x/a) + \sqrt{(a^2 - x^2)}$
Arc Cos (x/a)	$-1 / \sqrt{(a^2 - x^2)}$	$x \text{ Arc Cos } (x/a) - \sqrt{(a^2 - x^2)}$
Arc Tan (x/a)	$a / (a^2 + x^2)$	$x \text{ Arc Tan } (x/a) - a \ln \sqrt{(a^2 + x^2)}$
e^{ax}	$a e^{ax}$	$(1 / a) e^{ax}$
a^x	$a^x \ln (a)$	$a^x / [\ln (a)]$
ln (ax)	1 / x	x [ln (ax) – 1]
$\text{Log}_a x$	$(1/x) \text{ Log}_a e$	$x \text{ Log}_a (x / e)$
Sinh x	Cosh x	Cosh x
Cosh x	Sinh x	Sinh x
Tanh x	$\text{Sech}^2 x$	ln (Cosh x)
Arc Sinh (x/a)	$1 / \sqrt{(a^2 + x^2)}$	$x \text{ Arc Sinh } (x/a) - \sqrt{(a^2 + x^2)}$
Arc Cosh (x/a)	$1 / \sqrt{(x^2 - a^2)}$	$x \text{ Arc Cosh } (x/a) - \sqrt{(x^2 - a^2)}$
Arc Tanh (x/a)	$a / (a^2 - x^2)$	$x \text{ Arc Tanh } (x/a) + a \ln \sqrt{(a^2 - x^2)}$

Functions of Time and other variables

Velocity $v = dx/dt$

Acceleration = $d^2x/dt^2 = v \, dv/dx$

Functions of two or more variables

$V = F(x,y,z)$

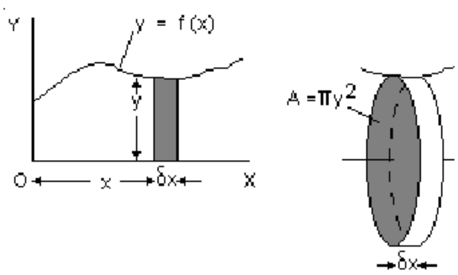
$\delta V = (\partial V/\partial x) \, \delta x + (\partial V/\partial y) \, \delta y + (\partial V/\partial z) \, \delta z$

where $\partial V/\partial x$ means the differential of V with respect to x while y and z are kept constant.

Areas and Volumes

Cones and Pyramids   Volume = (1/3) Base Area x Height

Volume of Revolution,
ie volume enclosed by rotating a curve about the x axis
Volume = $\int \pi y^2 \, dx$



Spheres   Surface Area = $4 \pi R^2$ = Curved area of enclosing cylinder
Volume = $4/3 \pi R^3$

Maxima and Minima

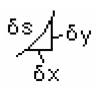
y = F(x) is a Maximum when $dy/dx = 0$ and d^2y/dx^2 is negative
y = F(x) is a Minimum when $dy/dx = 0$ and d^2y/dx^2 is positive
y = F(x) is a point of inflection when $dy/dx = 0$ and $d^2y/dx^2 = 0$

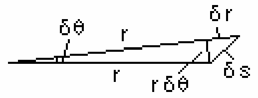
Maxima and Minima where F = f(x,y)
The conditions for a **maximum point** are;
$\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ and $[\partial^2 F/\partial x^2][\partial^2 F/\partial y^2] > [\partial^2 F/\partial x\, \partial y]^2$ and $\partial^2 F/\partial x^2$ is negative
And the conditions for a **minimum point** are;
$\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ and $[\partial^2 F/\partial x^2][\partial^2 F/\partial y^2] > [\partial^2 F/\partial x\, \partial y]^2$ and $\partial^2 F/\partial x^2$ is positive

If $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ and $[\partial^2 F/\partial x^2][\partial^2 F/\partial y^2] < [\partial^2 F/\partial x\, \partial y]^2$ then the point is a **Saddle Point**

Graphs

Length of Arc $s = \int \sqrt{[1 + (dy/dx)^2]}dx = \int \sqrt{[r^2 + (dr/d\theta)^2]}d\theta$





Radius of Curvature $\rho = [1 + (dy/dx)^2]^{3/2} / (d^2y/dx^2)$

Vectors

Definitions;
Scalar has Magnitude but not Direction
Vector has magnitude and Direction
The Operator j rotates a vector 90^0 anticlockwise, j^2 **V** = – **V** , hence $j = \sqrt{(-1)} = i$, is one solution
The Operator h rotates a vector 120^0 anticlockwise, hence h^3 **V** = **V** and $(1 + h + h^2)$ **V** = 0

I, **j** and **k** are three vectors mutually at right angles each length one unit.
Convention for axes. Shake Hands, Right Hand, Fingers point as **i** , Palm points as **j** , Thumb points as **k**

Matrix notation of a Vector. $| a_i \; a_j \; a_k |$ means Vector a_i **i** + a_j **j** + a_k **k**

IF **V** = V_x **i** + V_y **j** + V_z **k** then $V = \sqrt{[V_x^2 + V_y^2 + V_z^2]}$
Let θ be the angle between two vectors **V** = V_x **i** + V_y **j** + V_z **k** and **U** = U_x **i** + U_y **j** + U_z **k**
By definition, **V • U** = $V U \cos\theta$ hence **V • U** = $V_x U_x + V_y U_y + V_z U_z$ and **V • U** is a Scalar
and $\cos\theta = [V_x U_x + V_y U_y + V_z U_z] / \sqrt{[\{V_x^2 + V_y^2 + V_z^2\}\{U_x^2 + U_y^2 + U_z^2\}]}$
V and **U** are orthogonal if $V_x U_x + V_y U_y + V_z U_z = 0$
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ where α, β and γ are the angles between a vector and each axis.
By definition, **V ✕ U** = $V U \sin\theta$ **a** where **a** is a unit vector orthogonal to **V** and **U** hence **V ✕ U** is a Vector

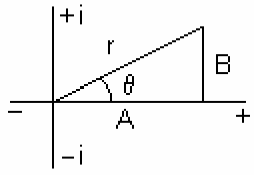
V ✕ U = the determinant
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ V_x & V_y & V_z \\ U_x & U_y & U_z \end{vmatrix}$$

If **A**, **B** and **C** define three adjacent edges of a parallelepiped
then Volume = **A ✕ B • C** = **B ✕ C • A** = **C ✕ A • B**

If a scalar value F is assigned to all points in a three dimensional volume, then by definition, Grad F (written ΔF) at any point is a Vector normal to the surface which connects the point to adjacent points which have the same value of F. ΔF has the magnitude equal to the differential of F with respect to distance in this direction

If a Vector **F** is assigned to all points in a 3D volume, then its differential is a Vector.
Div **F** is defined as $\partial \mathbf{F}/\partial x \bullet \mathbf{i} + \partial \mathbf{F}/\partial y \bullet \mathbf{j} + \partial \mathbf{F}/\partial z \bullet \mathbf{k} = \Delta \bullet \mathbf{F}$ and is a scalar.
and Curl **F** is defined as $\partial \mathbf{F}/\partial x \times \mathbf{i} + \partial \mathbf{F}/\partial y \times \mathbf{j} + \partial \mathbf{F}/\partial z \times \mathbf{k} = \Delta \times \mathbf{F}$ and is a vector.

Argand Diagram



The Complex Number A + iB can be represented as a Vector A + jB
$A + iB = r[\cos\theta + i\sin\theta] = r\,e^{i\theta}$
where $r = \sqrt{(A^2 + B^2)}$ and θ = Arc Tan (B/A)
Thus $[A + iB]^n = r^n\,e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)]$
Use for Multiplication or Division by Complex Numbers (and Vectors)
$\cos(2\pi n + \theta) + i\sin(2\pi n + \theta) = \cos\theta + i\sin\theta$ where n is any integer
Do not confuse the operators i or j in an Argand diagram with the unit vectors **i** and **j**.
(Operator j)^2 = –1 but (unit vector **j**)^2 = 1

Differential Equations

Definitions;
Ordinary or Partial (2 or more variables)
Order, if highest derivative is d^ny/dx^n Order is n
Arbitrary Constants. Solution has as many arbitrary constants as the Order. Evaluate by initial or final conditions.
Degree is the Index of highest derivative when rationalised
PI is the Particular Integral
CF is the Complementary Function
Complete Primitive = PI + CF
Singular Solution is an isolated solution
Linear Differential Equation has each term a Differential of y, all Degree one with Coefficient a function of x

(i) Solution of a Linear Differential Equation
Put $y = a_0 + a_1 x + a_2 x^2/2! + a_3 x^3/3! + \dots + a_r x^r/r! + \dots$
Check the answer has enough arbitrary constants

(ii) Exact Equations (first order) Mdx + Ndy = 0 This can be integrated immediately if $\partial N/\partial x = \partial M/\partial y$

(iii) Separate the variables to get $P(x)\,dx = Q(y)\,dy$ eg if $f(x)\,dy/dx = a$ then $y = \int[a/f(x)]dx + c$

(iv) Homogeneous Equations $dy/dx = f(y/x)$ Put $y = vx$

(v) Linear first order $dy/dx + P(x)\,y = Q(x)$ where P(x) and Q(x) are any function of x
multiply by integrating factor $R = e^{\int P\,dx}$

(vi) Linear, constant coefficients $F(D)y = f(x)$ for example $7D^2(y) - 3D(y) + 9y = 2 + 3x$
CF Solve F(D)y = 0 by substituting $y = A\,e^{ax} + Be^{bx}$ + etc where A, B, etc are arbitrary constants
Special cases (a) a and b conjugate pair $p \pm iq$, $y = e^{px}[A\cos(qx) + B\sin(qx)]$
(b) a = b, $y = A\,e^{ax} + Bxe^{bx}$
PI Find one solution to F(D) = f(x) and add to the CF to get the complete solution
Examples
(a) $f(x) = k_0 + k_1x + k_2x^2$ + etc Put $y = a_0 + a_1x + a_2x^2$ etc. Substitute and equate coefficients of x
(b) f(x) = k sin x or k cos x Put $y = a_1 \sin x + a_2 \cos x$ or take real (or complex) part of $y = a\,e^{ibx}$

(vii) Solution by Laplace Transform solves for f(t) and at the same time evaluates the arbitrary constants
This method is useful for evaluating the response of a control system
If $f(t) = A\,t^n\,e^{-at}$ then Laplace Transform $F(s) = A\,n!/(a + s)^{n+1}$
If $f(t) = A\,t^n \sin\omega t$ or $A\,t^n \cos\omega t$ then F(s) = Real or Complex part of $A\,n!/(s - j\omega)^{n+1}$
Laplace Transform of $d/dt\,[f(t)] = s\,F(s) - f(0)$
Laplace Transform of $d^2/dt^2\,[f(t)] = s^2\,F(s) - sf(0) - d/dt[f(0)]$
Laplace Transform of $\int f(t)\,dt = (1/s)\,F(s)$

(viii) $d^2y/dx^2 = -A\,y$ This is SHM. Solve by multiplying by the integrating factor $2\,dy/dx$

(ix) $x^2\,d^2y/dx^2 + x\,dy/dx + (x^2 - n^2)\,y = 0$
where n = 0. 1, 2, 3, 4, ... etc or n = 1/2, 1/3, 1/4, ... etc
This is Bessell's Eqtn.
Solution is $y = A\,J_n(x) + B\,Y_n(x)$ where A and B are arbitrary constants

$J_n(x) = \Sigma [\{(-1)^s (x/2)^{2s+n} \} / \{ \Gamma(s + n + 1) s! \}]$ from s = 0 to infinity
For +ive integers $\Gamma(x) = (x - 1)!$ for other values, $\Gamma(x) = \int t^{(x-1)} e^{-t} dt$ from t = 0 to ∞
$Y_n(x) = [\cos n\pi \; J_n(x) - J_{-n}(x)] / \sin n\pi$
$d^2y/dx^2 + x y = 0$ can be converted to Bessell's Eqtn by substitution

Fourier Series

Any cyclic function y = F(x) can be converted to a series of the form

$y = c_0 + a_1 \cos x + a_2 \cos 2x + ... a_n \cos nx + ...$
$+ b_1 \sin x + b_2 \sin 2x + ... b_n \sin nx + ...$

where

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} y \, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y \cos(nx) \, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y \sin(nx) \, dx$$

Summary (Part 2 Applied)

Mechanics

Constant acceleration equations
$v = u + at$ $s = (½)(u + v) t$ $s = ut + (½) at^2$ $v^2 = u^2 + 2as$
Gravitational Force $F = G M_1 M_2 / d^2$
Moment of Inertia $I = \int x^2 dm$

Newton's Laws (summarised)
(i) A body moves in a straight line unless acted on by a force
(ii) $P = ma$ and $C = I \, d\omega/dt = I \, d^2\theta/dt^2$
(iii) Action and Reaction are equal and opposite

Conservation of Energy
Work done = $F x = C \theta$
Kinetic Energy = $(½) m v^2$ = $(½) I \omega^2$
Potential Energy = $m g h$

Conservation of Momentum
Momentum = $m v$ Angular momentum = $I \omega$

(u before, v after collision) $m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$
$v_1 - v_2 = - e (u_1 - u_2)$

Body moving in a curve acceleration = $v^2/R = R \omega^2$

Two Dimension Forces in equilibrium Resultant Force in two directions = 0
Plus Couple about any one point = 0

Three Forces in equilibrium are co-planar and either meet at a point or are parallel

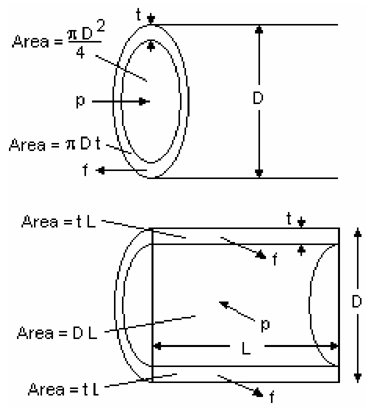
Friction Force = μN under gravity $F = \mu mg$

Simple Harmonic Motion (SHM) $d^2x/dt^2 = - k x$
hence $x = a \sin \omega t + b \cos \omega t$ where $\omega = \sqrt{k}$
T the Time for one cycle (ie the Period) is given by $\omega T = 2\pi$, hence Period = $2\pi / \sqrt{k}$

Capstan $P_2 = P_1 e^{\mu\theta}$

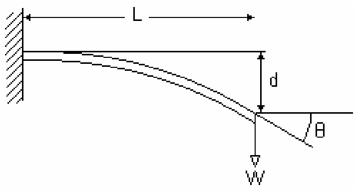
Structures Stress = p/A Strain = x / L E = Stress/Strain

Beam carrying a load $p / y = E / R = M / I$
where
p is stress at distance y from the Neutral Axis
E is Young's Modulus
R is radius of curvature
M is the bending moment
I is the 2nd moment of area about the Neutral Axis



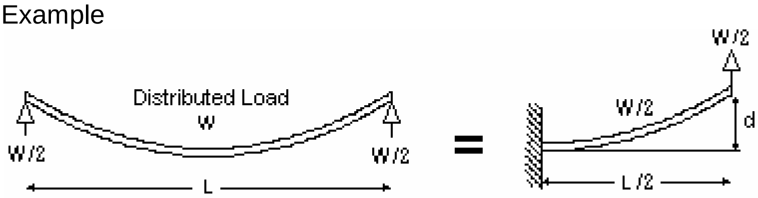
Hoop stress = $pD / 2t$
Longitudinal stress = $pD / 4t$
where p is pressure, D is outside dia, t is wall thickness
Use the outside diameter to allow for radial compression in the shell

Cantilever Beam



Moment M at L	$d = ML^2 /(2EI)$	$\theta = ML / (EI)$
Load W at L	$d = WL^3 /(3EI)$	$\theta = WL^2 / (2EI)$
Distributed Load W	$d = WL^3 /(8EI)$	$\theta = WL^2 / (6EI)$

Example



$d = (W/2)(L/2)^3 /(3EI) – (W/2)(L/2)^3 /(8EI) = 5 W L^3 /(384 EI)$

Suspension Bridge Parabolic $y = w x^2 /(2 F)$
Hanging chain Catenary $y = c [\cosh (x/c) – 1]$

Gyroscopes $\mathbf{C} = \omega \times \mathbf{M}$
Where **C** is a couple expressed as a corkscrew vector
ω is the angular velocity of precession expressed as a corkscrew vector
M is the angular momentum of the flywheel expressed as a corkscrew vector